McGee on Open-ended Schemas

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Abstract

A mathematical theory *T* is categorical if, and only if, any two models of *T* are isomorphic. If *T* is categorical, it can be shown to be semantically complete: for every sentence φ in the language of *T*, either φ follows semantically from *T* or $\neg\varphi$ does. For this reason some authors maintain that categoricity theorems are philosophically significant: they support the realist thesis that mathematical statements have determinate truth-values. Second-order arithmetic (PA^2) is a case in hand: it can be shown to be categorical and semantically complete. The status of second-order logic is a controversial issue, however. Worries about the purported set-theoretic nature and ontological commitments of second-order logic have been influential in the debate.

Recently, a number of authors – most notably Vann McGee and Shaughan Lavine – have argued that one can get some of the advantages of second-order axiomatisations – categoricity, in particular – while walking free of the standard objections against second-order logic. In so arguing they appeal to the notion of an open-ended schema. In "open-ended arithmetic", induction holds not only for formulas in the current language, but for formulas in *any* extension of that language. Both McGee and Lavine stress that this is not merely a variant of second-order logic. In this paper, we discuss the defensibility of this claim on McGee's part. McGee's account involves a meta-theoretic rule that provides information about extensions of the language are legitimate. We argue that in the presence of this rule open-ended schemas are as ontologically suspect as standard second-order quantification.

1 Technical Notions

Let us start by rehearsing some standard definitions.

Define the notion of a theory as follows:

Definition 1 (Theory) Let *A* be the axioms of a theory *T*. The theory of *T* is the closure of *A* under semantic consequence, i.e. $T = \{\varphi : A \vDash \varphi\}$.

For example, if *T* is Peano arithmetic, the theory of *T* is the axioms of Peano arithmetic together with all their semantic consequences. (Obviously, the axioms *A* of a theory *T* will always be included in the theory since $\varphi \vDash \varphi$ for any φ.)[1]

Define the notion of categoricity as follows:

Definition 2 (Categoricity) A theory *T* is *categorical* if, and only if, any two models of *T* are isomorphic.

The notions of a model and isomorphism relied on in Definition 2 are completely standard. An interpretation of a theory *T* is a structure $I = \langle d, i \rangle$, where *d* is a non-empty set (the *domain* of *T*) and *i* is an interpretation function that assigns referents to the items of the language of *T*. A model of *T* is an interpretation which makes the axioms of *T* true (and so, all their semantic consequences). An isomorphism between two models *M* and *N* is a bijection between *M* and *N* that

[1] Due to soundness and (deductive) completeness (cf. footnote 6), it will not matter in a first-order setting whether we speak in terms of semantic or syntactic, or deductive, consequence.

preserves structure.[2]

Provided that T is categorical, it is straightforward to show that T is also semantically complete:

Definition 3 (Semantic completeness) A theory T is *semantically complete* if, and only if, for any sentence φ in the language of T, either $T \vDash \varphi$ or $T \vDash \neg\varphi$.

That is, a theory T is semantically complete if, and only if, for any sentence φ in the language of T, a model of T makes either φ or its negation true.[3]

The definition of semantic consequence involves the notion of truth-in-a-model. A sentence φ is said to follow semantically from a set of sentences Γ if, and only if, every model in which every member of Γ is true is also a model in which φ is true.

Now, define the notions of determinate truth and determinate falsity as follows:

Definition 4 (Determinate truth) Let T be a theory, and let $\mathcal{M}$ be the class of

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2 This is merely an informal gloss on the notion of an isomorphism. However, for our present purposes it will suffice. For a full definition of the notion, cf., e.g., (Enderton, 1972).

3 Categoricity implies semantic completeness:
Suppose that theory T is categorical. It can be so either trivially – by having no models – or by having models that are all isomorphic. In the trivial case, there is some sentence ψ in the language of T such that $A \vDash \psi \wedge \neg\psi$. But $\psi \wedge \neg\psi \vDash \varphi$ for any sentence φ, and hence, $A \vDash \varphi$, as well as $A \vDash \neg\varphi$. Thus, for any φ, $A \vDash \varphi$ or $A \vDash \neg\varphi$. For the non-trivial case, consider an arbitrary model M of T. Since we are working with classical models, any sentence φ of the language of T is either true or false in M. There are two cases to consider. First, assume that φ is true in M. It is a result of standard model theory that isomorphic structures agree on the truth values they confer upon sentences. So, φ is true in any model of T. Clearly, the axioms are true in any model of T – so whenever A is true, φ is true, whence $A \vDash \varphi$. The second case – where φ is false in M – is similar. So, for any φ, either $A \vDash \varphi$ or $A \vDash \neg\varphi$, as required.

models of *T*. Then a formula φ in the language of *T* is *determinately true* if, and only if, φ is true in every member of $\mathscr{M}$, and φ is *determinately false* if, and only if, ¬φ is true in every member of $\mathscr{M}$.

The notion of determinate truth incorporates the notion of truth-in-a-model. As should be clear, a sentence's being determinately true requires more than it's being true-in-a-model. The former requires that the sentence be true in *all* models of the relevant class of models, while the latter only requires that it be true in one such model. To see this consider the continuum hypothesis (CH) and ordinary first-order Zermelo-Fraenkel set theory (ZF^1). By the results of Gödel and Cohen, there are models of ZF^1 in which CH is true, and there are also models in which CH fails.[4] Consequently, given the class of models of ZF^1, CH is not determinately true (and not determinately false either).

What is crucial to note for our current purposes is that, if *T* is a categorical theory, every sentence φ in the language of *T* has a determinate truth-value. That is, if *T* is categorical, every sentence φ is either true in all models of *T* or false in all models of *T*. This is because categoricity implies semantic completeness, and semantic completeness implies determinacy of truth-value. To see that semantic completeness implies determinacy of truth-value, suppose that *T* is semantically complete. In that case, for any sentence φ of the language of *T*, either $A \vDash \varphi$ or $A \vDash \neg\varphi$ (where '*A*' refers to the set of *T*-axioms). Suppose that $A \vDash \varphi$, and let $\mathscr{M}$ be the class of models of *T*. Every member of *A* is true in any member of $\mathscr{M}$. By our assumption (and the definition of semantic

[4] (Gödel, 1938), (Cohen, 1963–64).

consequence), φ is true in every member of $\mathcal{M}$, and hence, determinately true by Definition 4. The case for $A \vDash \neg\varphi$ is similar.

2 Second-order theories and truth-value realism

So far, so good. But why should the technical notions presented in section 1 be thought to carry any philosophical significance? The answer is that they stand in an intimate conceptual relationship to truth-value realism, a substantial philosophical thesis. A truth-value realist with respect to a given domain of discourse *D* is someone who embraces the following thesis:

(TVR) Every sentence of D is either determinately true or determinately false.

Categoricity implies semantic completeness, and semantic completeness implies determinacy of truth-value. So, categoricity seems to give the realist exactly what she is after: a way of supporting (TVR).[5] There are critics – see, for instance, (Field, 1994) and (Field, 2001a) – who question that these conceptual links obtain. However, the objective here is to discuss the claimed merits of open-ended schemas over standard second-order quantification, on the assumption that these approaches deliver categoricity and determinacy of truth-value. Thus, for the purposes of this paper, it will be assumed that the conceptual links in question do obtain.

Let us focus on truth-value realism with respect to arithmetic and set theory.

[5] What is here referred to as “truth-value realism” is referred to as “realism in truth-value” and “realism about truth” in respectively (Shapiro, 1991) and (McGee, 1997).

To see how categoricity can assist the truth-value realist we need to ask what theories can be shown to be categorical. Due to the meta-theoretic properties of first-order theories, some popular candidates are ruled out on principled grounds:

- First-order Peano arithmetic (PA^1) is not categorical.
- First-order real analysis (RA^1) is not categorical.
- First-order Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC^1) is not categorical.

These candidates are ruled out because of the following, well-known result:

Gödel's first incompleteness theorem:

If T is consistent, has a recursive set of axioms, and is powerful enough to express elementary arithmetic, then there is a sentence ψ of the language of T such that neither ψ nor $\neg\psi$ is provable in T.

Gödel's theorem gives a cogent reason why PA^1, RA^1, and ZFC^1 – indeed, any reasonably strong first-order theory – cannot do the technical work needed by the truth-value realist. To see this first observe that, by Gödel's theorem, for any reasonably strong first-order theory T which is consistent and has a recursive set of axioms, there is a sentence ψ such that neither $T \vdash \psi$ nor $T \vdash \neg\psi$. Due to soundness and (deductive) completeness[6], the deductive and semantic consequence relations coincide in the first-order case, so there is a sentence ψ such that neither $T \vDash \psi$ nor $T \vDash \neg\psi$ In other words, T is semantically *in*complete, and so, there is some sentence in the language of T that does not have a determinate truth-value.

Second-order theories, on the other hand, do much better here. The language of second-order logic is obtained from the language of first-order logic by adding predicate variables, function variables, and second-order quantifiers to bind these. Rules are added to specify the syntactic behaviour of these items. This is standard material, so we will not dwell on it here.[7]

The most common way of doing semantics for second-order logic is to use set theory (or class theory). There are two common ways of doing the semantics:

Standard semantics:

A model of second-order logic with standard semantics is a structure $I = \langle d, i \rangle$, where d is a non-empty set (the domain) and i the interpretation function of I. First-order variables are assigned members of d as their denotation. Let d^n be the set of n-tuples on d. Every n-place predicate variable is assigned a member of its denotation, and n-place function-variables denote functions from d^n to d (for $d^2 = d \times d$, etc.). The

6 The notion of deductive completeness relevant here is different from semantic completeness: it refers to the converse of soundness. A theory T is deductively complete if, and only if, for any sentence φ and set of sentences Γ in the language of T: if φ is true in all models that make all sentences in Γ true, then φ is derivable from Γ in the deductive system of T (or formally: $\Gamma \vDash \varphi \Rightarrow \Gamma \vdash \varphi$).

7 For details, see (Shapiro, 1991, chapter 3), following (Church, 1956, chapter V).

assignment of denotation to *n*-place predicate variables is a function from the set of *n*-place predicate variables to the set of all *n*-tuples on *d*, i.e. the second-order variables are assigned semantic values on the full power set of the domain.

Henkin semantics:

A model of second-order logic with Henkin semantics is a structure I_H = <*d*, *D*, *F*, *i*>. As before, *d* and *i* are the domain and the interpretation function of the model, respectively. *D* is a fixed collection of relations on *d*, and *F* is a fixed collection of functions on *d*. For every *n*, *D*(*n*) is a non-empty subset of the power set of d^n. For every *n*, *F*(*n*) is a non-empty set of functions from d^n to *d*.

Every standard model is a Henkin model, but not vice versa. As indicated, standard models are models in which the second-order variables range over the full power set of the first-order domain. In some Henkin models, the second-order variables take the same range. However, there are Henkin models – *proper* Henkin models – in which the range of the second-order variables is restricted so that it is less than the full power set of the first-order domain.

This difference is crucial in relation to categoricity. Second-order logic with Henkin semantics has the same meta-theoretic properties as first-order logic, i.e. it is (deductively) complete, compact, and the Löwenheim-Skolem theorems hold.[8] This means that the argument based on Gödel's theorem transfers to

[8] Completeness has been stated in footnote 6 above. Compactness and the LS theorems

second-order theories where the background logic is taken to be second-order logic with Henkin semantics. Consequently, second-order logic with Henkin semantics offers no hope of categoricity, semantic completeness – and, with these, determinacy of truth-value.

However, matters are different for second-order theories where the background logic is taken to be second-order logic with *standard* semantics. In particular, the following results can be established:

- Second-order Peano arithmetic (PA^2) is categorical.[9]
- Second-order real analysis (RA^2) is categorical.[10]
- Second-order ZFC (ZFC^2) is quasi-categorical, i.e. for any two models of the theory, one is isomorphic to an initial segment of the other.[11]
- Second-order ZFC with urelements ($ZFCU^2$) with the Urelement Set Axiom is categorical.[12]

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read as follows (in one formulation):

- **Compactness Theorem:** Let Γ be a set of formulae. Then Γ has a model if, and only if, every finite subset of Γ has a model.
- **Downward LS Theorem:** Let Γ be a set of formulae. If Γ has an infinite model, then Γ has a model whose domain is at most denumerable.
- **Upward LS Theorem:** Let Γ be a set of formulae. If Γ has an infinite model, then Γ has a model of every infinite cardinality.

9 Originally established by Dedekind. For a proof, see (Shapiro, 1991, 82–83).

10 See (Shapiro, 1991, 84).

11 This result is a variation of a result of (Zermelo, 1930). In ZFC, whether first- or second-order, there are no non-sets in the domain. Zermelo's result is more general in that his framework allows for urelements, i.e. non-sets. What he showed was that, for any two models of ZFC^2 with the same base, one is isomorphic to an initial segment of the other. The theorem cited for ZFC^2 is the special case of Zermelo's result where there are no urelements.

Shapiro suggests including a general choice principle in the deductive system of second-order logic rather than the weaker comprehension schema for functions; the set-theoretic version of choice follows from this general choice principle (Shapiro, 1991, 67). Thus, when dealing with second-order set theory, we could drop the axiom of choice from the list of axioms as we get it "for free"' from the background logic.

12 (McGee, 1997). The Urelement Set Axioms states that all the non-sets form a set. It should also be noted that in order for the proof of categoricity to go through it needs to

So, it would seem that in at least some cases the realist can get what she wants by adopting a certain second-order theory where the background logic is taken to be second-order logic with standard semantics. But can she really?

3 Objections to second-order logic

The status of second-order logic is a hotly debated topic. A common objection to the claim that categoricity results support determinacy of truth-value is that the employment of the resources of second-order logic (with standard semantics) is illicit. The complaints leveled against second-order logic are numerous (and some related):

- The semantic consequence relation of second-order logic is epistemically untractable.
- The range of second-order variables is the full powerset of the first-order domain, i.e. the set of arbitrary subsets of the first-order domain. However, the notion of an arbitrary subset is obscure when the domain is infinite, as is the case for arithmetic and set theory.
- Second-order logic carries set-theoretic content, but logic proper should not do so.

We will not dwell on the details of these complaints here.[13] However, it is worth touching on W.V. Quine's worry about the "staggering existential assumptions" of second-order logic – if only very briefly. It is worth doing so, because McGee

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be assumed that the range of the first-order quantifiers is unrestricted.

[13] A non-exhaustive list of references will have to do: (Cutler, 1997), (Jané, 1993), (Jané, 2005), (Field, 1994), (Field, 2001a), (Resnik, 1988), (Wagner, 1987), (Weston, 1976).

claims that open-ended schemas do better than second-order quantification with respect to this worry, and this is the claim that will be contested later in this paper.

In a much discussed passage from his *Philosophy of Logic*, Quine writes:

> Followers of Hilbert have continued to quantify predicate letters, obtaining what they call a higher-order predicate calculus. The values of these variables are in effect sets; and this way of presenting set theory gives it a deceptive resemblance to logic. One is apt to feel that no abrupt addition to the ordinary logic of quantification has been made; just some more quantifiers governing predicate letters already present. In order to appreciate how deceptive this line can be, consider the hypothesis '$(\exists y)(x)(x \in x \,.\leftrightarrow Fx)$'. It assumes a set $\{x : Fx\}$ determined by an open sentence in the role of 'Fx'. This is the central hypothesis of set theory, and the one that has to be restrained in one way or another to avoid the paradoxes. This hypothesis itself falls dangerously out of sight in the so-called higher-order predicate calculus. It becomes '$(\exists \Gamma)(x)(Gx \leftrightarrow Fx)$', and thus evidently follows from the genuinely logical triviality '$(x)(Fx \leftrightarrow Fx)$' by an elementary logical inference. Set theory's staggering existential assumptions are cunningly hidden now in the tacit shift from schematic predicate letter to quantifiable set variable.[14]

The quoted passage appears at the end of the section entitled 'Set theory in sheep's clothing'. As the section title suggests, Quine holds that second-order logic (or higher-order logic more generally) is set theory cleverly disguised as logic. His complaint must be something like this:

[14] (Quine, 1970, 68).

Second-order quantification is quantification into predicate position. The second-order existential quantifier, thus, must commit us to some kind of universal.[15] The only agreeable universals are sets, since at least they are extensional, in contrast to attributes, properties, relations, and the like. So, in the best case scenario, second-order logic carries ontological commitment to sets. Sets are respectable entities for Quine, so a commitment to them would not be bad as such. Second-order logic masks this ontological commitment, however: the set-membership symbol, '∈', disappears. Thus, the use of second-order logic is intellectually dishonest; using set theory explicitly is the way to go.

Contemporary (and more sophisticated) versions of the set-theory allegation against second-order logic usually refer to the deep entanglement of the consequence relation of its standard semantics with set theory.[16] The verdict remains the same, however: second-order logic has set-theoretic content.

4 McGee on schemas and open-endedness

In recent contributions to the literature, Vann McGee has advocated the view that open-ended schemas offer a way to obtain categoricity results for central mathematical theories without raising the objections standardly launched against second-order quantification.[17] Worries about the ontological commitments of second-order logic do not apply to open-ended schemas according to McGee, since second-order quantifiers and variables are absent. In due course, it will be argued that one cannot escape these worries in the way

[15] This is the step that is usually denied by Quine's critics – see e.g. (Boolos, 1984), (Boolos, 1985), (Rayo and Yablo, 2001, 79–80), (Wright, 1983, 132–133).

[16] For references, see footnote 13 above.

[17] See (McGee, 1997). In (McGee, 2000), McGee endorses the view that logical rules are to be understood in an open-ended fashion.

McGee suggests.[18]

To see what the notion of an open-ended schema amounts to consider first-order Peano arithmetic (PA^1). This theory contains a schema, *viz*. the axiom schema of induction:

$$(\Phi(\mathbf{0}) \wedge \forall x(\Phi(x) \rightarrow \Phi(sx))) \rightarrow \forall x \Phi(x)$$

i.e. for any Φ, if Φ holds of zero and Φ holds of the successor of any natural number provided Φ holds of the natural number itself, then Φ holds of all natural numbers.

The schema itself, it should be noted, is not part of PA^1, i.e. it is not part of the object theory. It cannot be written down as a single sentence in the language. However, what *is* part of the theory is each of the individual axioms obtained by substituting an open formula into the schema.

In some sense, the schema gives us a fair amount of freedom. It allows us to substitute *any* formula of the language of first-order arithmetic to generate an individual axiom. However, the axiom schema does come with a substantive, principled restriction. The restriction is that any formula Φ we are allowed to substitute into the schema has to be a formula *in the language of first-order arithmetic*. That is, the induction schema of PA^1 is restricted to a specific language – namely the language of first-order logic supplemented by '**0**' and '*s*'.

Open-ended schema arithmetic is obtained from first-order arithmetic by

[18] Shaughan Lavine has developed an approach similar to that of McGee (see (Lavine, 1994), (Lavine, 1999), and (Lavine, 2006)). However, attention will be restricted to McGee's notion of open-endedness in this paper.

dropping this restriction, i.e. by taking the induction schema to hold no matter how the language is extended. New individual names can be added to the language as well as new predicate letters. In this sense the schema is *open-ended*. According to McGee, what we learn when we learn the language of arithmetic is, rationally reconstructed, open-ended schema arithmetic. The following passages support this interpretation of McGee:

> Rationally reconstructed, what we are taught when we learn the language of arithmetic is the Peano axioms, so understood that we can substitute into the Induction Axiom any open sentence we like. We can substitute any open sentence of English. Moreover, if we extend present-day English by adjoining additional vocabulary, we expect to be able to substitute in any open sentence of the extended language [...]. Our understanding of the language of arithmetic is such that we anticipate that the Induction Axiom Schema, like the laws of logic, will persist through all such changes. There is no single set of first-order axioms that fully expresses what we learn about the meaning of arithmetical notation when we learn the Induction Axiom Schema, since we are always capable of generating new Induction Axioms by expanding the language.[19]

> If someone responded to the introduction of a new theoretical term into her vocabulary by asking, “I wonder whether mathematical induction still holds for open sentences containing the new term?” we would say that she didn’t properly understand the way we use arithmetical vocabulary, because she hadn’t properly grasped the full range of application of the Induction Axiom Schema.[20]

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[19] (McGee, 1997, 58).
[20] (McGee, 1997, 59).

On McGee's view, then, open-ended schema arithmetic succeeds where first-order arithmetic fails. First-order arithmetic fails to fully capture at least one crucial aspect of our mastery of arithmetic, namely that we understand that the induction schema remains in force when we extend the language by adding new names (of individuals) and new predicate letters. Open-ended schema arithmetic also does better than second-order arithmetic. The source of concern with respect to the latter is the reliance on second-order quantification – one of the major issues being the ontological commitment of this kind of quantification, as seen above. However, according to McGee, the issue of ontological commitment vanishes once the switch is made to open-ended schemas:

> One worries that second-order quantification entangles us in unsavory ontological commitments. There is no such worry about the rule. Adoption of a rule permitting us to assert a sentence φ only commits us to the ontological commitments of φ, whatever they are. So adoption of a rule permitting the assertion of Induction Axioms only commits us to the ontological commitments of Induction Axioms, and Induction Axioms are only committed to numbers.[21]

To be slightly more explicit than in the quote, the rule of which McGee speaks allows one to substitute any (legitimate) open sentence into the induction schema. What counts as a legitimate open sentence is not restricted to the language of first-order logic supplemented by '**0**' and '*s*', but is understood in an open-ended manner.

The usual way of doing semantics for second-order logic is to take the

[21] (McGee, 1997, 60).

semantic values of the second-order variables to be sets (or classes).[22] Endorsing the induction axiom of second-order arithmetic thus commits one to the existence of *sets* (or collections) of natural numbers.[23] In light of this, second-order quantification is thought to bring on "unsavory ontological commitments". However, according to McGee, for open-ended schemas the commitment to sets, or collections, is gone. In particular, individual axioms obtained by substituting an open sentence into the induction schema "are only committed to numbers". Schemas are "metaphysically benign"[24], but delivers exactly what the realist is after, i.e. categoricity – and (through semantic completeness) determinacy of truth value:

> This schematic version of the Categoricity Theorem gives the realist precisely what she wants: a system of symbolic representations that is simple and easy to use, yet demonstrably powerful enough to affix a determinate truth value to each mathematical sentence.[25]

There is no point in going into the details of the schematic versions of the proof of categoricity here. At the technical level, open-ended schemas achieve what they are supposed to.[26] However, contrary to McGee's contention, the open-ended schema approach faces significant philosophical problems. In order to see this, we will first provide an account of McGee's notion of open-endedness.

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22 See (Shapiro, 1991, section 3.3).

23 Note here that, when McGee speaks of an induction axiom being committed to only numbers, he is talking about a specific axiom obtained from a schema by substitution of an open sentence for Φ – which is different from the induction axiom of second-order arithmetic which is the single sentence: $(\forall X)((X\mathbf{0} \wedge \forall x(Xx \rightarrow Xsx)) \rightarrow \forall x\, Xx)$.

24 (McGee, 1997, 6).

25 (McGee, 1997, 62).

26 We refer the reader to (Lavine, 1999) and (McGee, 1997).

The notion, as we shall see, can be summarised in a rule concerning what individuals and collections of individuals are nameable. Once the rule has been stated, we will try to make trouble for it. This will be done in the next section.

McGee links open-endedness to what individuals and classes of individual are nameable. He writes:

> To say what individuals and classes of individuals the rules of our language permit us to name is easy: we are permitted to name anything at all. For any collection of individuals K there is a logically possible world – though perhaps not a theologically possible world – in which our practices in using English are just what they are in the actual world and in which *K* is the extension of the open sentence '*x* is blessed by God'. So the rules of our language permit the language to contain an open sentence whose extension is *K* [...]. This holds for any collection *K* whatever, whether or not we are psychologically capable of distinguishing the *K*s from the non-*K*s.[27]

What McGee says in the quote can be captured reasonably by the following rule:

McGee's Rule:

(1) Any individual is nameable. If, for a given individual, *L* does not already contain a name for it, such a name can be added to *L*.

(2) Any collection of individuals *C* is nameable, in the sense that there is an open sentence φ such φ holds exactly of the members of *C*. If *L*

[27] (McGee, 1997, 59).

does not contain all the predicates (or other expressions) occurring in φ, such items can be added to *L*.

This is a meta-theoretic, rather than object-theoretic, rule. It provides two kinds of information. First, it tells us how the object-language can be extended, and second, it tells us how new items of the language relate to the domain of the relevant theory. For instance, if we are dealing with arithmetic, the rule is not among the arithmetical axioms, but is instead part of the meta-theory. It tells us that any object in the domain, i.e. any number, is nameable, and that, for any collection of natural numbers, there is an open sentence Φ such that Φ holds exactly of the numbers in that collection. We are also told that, if the current language lacks the resources to facilitate these namings, the requisite items can be added to the language.

5 Against McGee

The aim of this section is to undermine McGee's claim that the open-ended schema approach is philosophically superior to standard second-order quantification. The basic point against McGee is simple: there is a sense in which McGee's Rule makes the open-ended schema logic equivalent to second-order logic with standard semantics. As a consequence, McGee cannot cash in on all the philosophical promises made on behalf of open-ended schemas. In particular, contrary to what he himself claims, they are all but metaphysically benign. As we shall see, open-ended schemas and second-order logic are very much on a par when it comes to ontological commitment.

The reason why open-ended schemas and second-order logic are on a par in terms of ontological commitment is that they are, in some sense, equivalent. Here is an argument for the equivalence (or rather what might be considered the controversial direction of the equivalence):

In second-order logic with standard semantics, the range of the second-order quantifiers is the full power set of the first-order domain. The second-order variables range over the arbitrary subsets of the first-order domain. Similarly, open sentences that can be substituted into open-ended schemas take their semantic value among, or are true of, the members of arbitrary subsets of the first-order domain. This is due to the second clause of McGee's Rule which tells us that any collection of individuals is nameable. Hence, for every instance of a second-order axiom, the open-ended schema delivers a corresponding individual axiom. To see this, consider some instance of the relevant (second-order) axiom and suppose that *C* – a collection of members of the first-order domain – is the semantic value of the second-order variable for this instance. Then McGee's Rule tells us that *C* is nameable in the sense that there is an open sentence φ such that φ is true exactly of the members of *C* (and that if items figure in φ that are not part of the "base language", then these items can just be added to the language). This open sentence can be substituted into the open-ended schema. For, after all, the whole point of taking schemas to be open-ended is to let them remain in force when the language is expanded.

In sum, whatever second-order axioms deliver, open-ended schemas will match. If it was ever mysterious how open-ended schemas get their potency to cater for categoricity results, this should be somewhat clearer now. What is also the case, however, is that open-ended schemas are philosophically suspicious if second-order quantification is – in particular in terms of ontological commitment. If so, the motivation for the open-ended schema approach is gone. It was, recall, the philosophical advantages of open-ended schemas that were supposed to tip the scales in their favour.

Let us offer some backing for our claim that open-ended schemas are just as ontologically extravagant as second-order quantification. In this argument we will follow Quinean orthodoxy and take it that to be is to be the value of a (bound) variable. Thus, it will be assumed that the ontological commitments of a given theory *T* are the values of its bound variables. There are, of course, those who question Quinean orthodoxy.[28] However, what is relevant to note here is that McGee does not seems to question it,[29] and so we allow ourselves to adopt this criterion of ontological commitment below.

As we have seen above, McGee argues that, unlike second-order quantification, endorsement of open-ended schemas does not commit one to the existence of sets, or classes, of objects in the first-order domain (numbers in the case of arithmetic).[30] However, as a little work will show, the presence of McGee's Rule ultimately brings on a commitment to such entities. Above it was noted that the rule in question is a meta-theoretic rule, and so, not part of the object-theory. Supposing that we are considering open-ended schema

[28] As Boolos put it nicely, "[t]he difficulty with the question [concerning ontological commitment] is that the ground rules for answering it appear to have been laid down, by Professor Quine." (Boolos, 1985, 76).

[29] (McGee, 1997, 60); Lavine also seems to take it on board: (Lavine, 1994, 227).

[30] We will not discuss the plural interpretation of second-order quantification in this paper. Boolos has shown how to interpret monadic second-order logic using plural devices and argues that so doing steers clear of the ontological commitments supposedly carried by second-order logic as it is standardly conceived (cf. (Boolos, 1984) and (Boolos, 1985)). The plural interpretation can be extended to *n*-adic second-order quantification using a pairing function, see (Burgess, Hazen, Lewis, 1991). See also (Rayo, 2002), (Rayo and Yablo, 2001), but also (Resnik, 1988) and (Linnebo, 2003). For an extension of the plural approach to n^{th}-order quantification, see (Rayo, 2006).

arithmetic, it should thus be granted that open-ended schema arithmetic is not committed to sets of natural numbers at the object-theory level. For no set of natural numbers is the semantic value of any of the variables of the theory. Open-ended schema arithmetic only has first-order quantifiers, and first-order variables take numbers as their semantic values.

However, let us switch focus to the meta-theory. When we move to the meta-theory, we encounter McGee's Rule. This rule explicitly talks about collections of members of the first-order domain: any collection of members of the first-order domain can be named. In other words, there is a quantifier in the meta-theory that ranges over collections of the first-order domain. McGee's rule thus carries a commitment to collections of members of the first-order domain. At the meta-theoretic level, then, McGee is committed to the existence of entities which, when figuring as values of object-theory variables (of the second-order theory), were deemed an "unsavory ontological commitment". There is something uncomforting about this. For pushing the commitment one level up does not make it disappear. It might be at one remove from where it "used to be" (in the second-order theory), but it is still there and is made no more a savory commitment when pushed into the meta-theory.

At the end of the day, what you are ontologically committed to includes not just the entities given by the ontological commitments of the object-theory. The commitments of the meta-theory must be included as well. The same goes for the meta-meta-theory. And so on. If a certain kind of ontological commitment is to be avoided, it had better be avoided altogether. That is, there should be no commitment of the relevant kind anywhere in the hierarchy. The reason that one is ontologically committed "all the way up" is that each theory in the hierarchy should be *true*.

Suppose that open-ended schema arithmetic is the object-theory, and suppose – with McGee – that this is our theory of arithmetic. Then we want this theory to be true. Assuming that it *is* true, the Quinean criterion of ontological commitment tells us that we are thus committed to the existence of whatever the values of the bound variables of the theory are. As noted above, open-ended schema arithmetic only has first-order quantifiers, and the values of the first-order variables are numbers. However, in addition to open-ended arithmetic – the object-theory – being true, we would also want the meta-theory to be true. Why? Because the open-ended schema of induction (as opposed to the individual axioms generated from it) and McGee's Rule are part of this theory rather than the object-theory. They are, however, an, not *the*, crucial part of whole approach.

The truth of McGee's Rule is needed (i) in order to make sense of the idea of an open-ended schema, and (ii) in order to generate enough (individual) induction axioms to go into the object-theory. As regards (i), McGee's Rule is not the only way to make sense of a schema that allows for substitution of formulas that go beyond the resources of a given language. However, presumably, McGee would maintain that it is the only way to make sense of a *genuinely* open-ended schema, or at the very least, his favourite or the most promising one. (And we can see no other that does the trick and but does not involve, in some way, second-order quantifiers.) As for (ii), the meta-theoretic rule that specifies what items can legitimately be added to the language must be powerful enough to support categoricity – at least if, like McGee (and Lavine), we are interested in categoricity with a view to supporting determinacy of truth-value.

But now there is trouble. For McGee's Rule involves a quantifier in the second clause that ranges over collections of the first-order domain, i.e. over collections of numbers. These collections are values of bound variables of the meta-theory, and thus, supposing that the rule in question is true – as McGee would have it – we are committed to the existence of sets, or collections, of natural numbers in the meta-theory. (Again, this is a consequence of adopting the Quinean criterion of ontological commitment.) Although the commitment to sets, or collections, of natural numbers is at one remove from where it was in the case of second-order arithmetic, it is still there. And moving it there makes it no less troublesome.

6 Conclusion

The following was been accomplished in the previous section: McGee claims that open-ended schemas are more innocuous than ordinary second-order quantification, particularly in terms of ontological commitment. It was argued that this is not the case. What was referred to as 'McGee's Rule' brings with it a meta-theoretic commitment to sets, or collections, of objects of the first-order domain. It was further argued that shifting the ontological commitment one level up does not make it any less unsavory from a philosophical perspective. An ontological commitment is an ontological commitment, whether it is here or there. Since the meta-theoretic level is so heavily involved in achieving the desired outcome of the schematic approach, this commitment goes on the score – it cannot be made to work without it.

The aim of our paper was not to defend full second-order logic and argue that it is superior to the schematic approach. Rather, we assumed the charge of ontological offensiveness against second-order logic for the sake of argument, since McGee (and Lavine) appears to agree with it. We then argued that, despite claims to the contrary, the schematic approach incurs the same “unsavory ontological commitments” that second-order logic allegedly has – if indeed it does. All that is needed to make a case against McGee is an argument to the effect that open-ended schemas do no better than second-order logic. This is exactly the kind of argument that we presented in this paper. Contrary to McGee’s contention, we thus conclude that open-ended schemas do not offer any advantage over standard second-order quantification with respect to the issue of ontological commitment.

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