Entitlement in mathematics

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Crispin Wright has recently introduced a non-evidential notion of warrant – *entitlement of cognitive project* – as a promising response to certain sceptical arguments, which have been subject to extensive discussion within mainstream epistemology. The central idea is that, for a given class of cognitive projects, there are certain basic propositions – entitlements – which one is warranted in trusting provided there is no sufficient reason to think them false. (See Wrigh [2].) The aim of this paper is to provide an account of the notion of entitlement of cognitive project and briefly discuss the question whether there is any work for the notion of entitlement to do within the philosophy of mathematics. Bearing in mind its applications in mainstream epistemology, it will be suggested that the notion can be used to formulate a response to certain kinds of scepticism which call into question the warrantability of (acceptances of) propositions that appear integral to mathematical theorizing in a given mathematical theory T – in particular, that T is consistent and that T's background logic is sound.

1 Cornerstones

Let us start with a piece of terminology:

(COR) A certain proposition – or a specific type of proposition – is a *cornerstone* for a given region of thought just in case the proposition (or type of proposition) is such that, if we had no warrant for it, we could not rationally claim warrant for any belief in a proposition of that region of thought.

Our thinking about the empirical world is an example of a region of thought. The cornerstones of this region include propositions that have been at the heart of the discussion of scepticism in mainstream epistemology – for instance, that I am not being deceived by an omnipotent, but evil demon or experiencing a vivid, coherent dream. Suppose that I had a warrant for neither of these propositions. Furthermore, suppose that I have a visual experience of what seems to be a tree in front of me, and that I form a belief that there is a tree in front of me on the basis of this experience. Can I rationally claim to have a warrant for the belief in question? It would seem not. Because the propositions that I am not being deceived by an omnipotent, but evil demon and that I am not experiencing a vivid, coherent dream concern something integral to my investigation, *viz.* whether the attendant circumstances are suitable for proper belief-formation.

Note that this is not to say that, absent a warrant for a cornerstone of a given region of thought, one cannot possess a warrant for belief in any proposition of that region. Rather, what the characterization of the notion of a cornerstone commits us to is the incompatibility of an absence of warrant for a cornerstone and a rational *claim* to hold a warranted belief in some proposition from the relevant region.

2 Mathematical cornerstones: consistency and soundness

Having given a bit of background, let us shift focus to the case that shall concern us here, the mathematical case. Regions of thought can be more or less general. The most general region would be mathematical thinking as such. However, here we shall proceed at a level of less generality by talking about theorizing within some specific mathematical theory T.

Consider a mathematical theory T. Let us use ZFC in its standard first-order formulation as an example throughout. What kind of propositions are candidate cornerstones of ZFC-theorizing? Here are two such candidates:

(CONSISTEMCY) ZFC is consistent.

(SOUNDNESS) The rules of inference of the background logic of ZFC are sound.

As usual, we take a set of sentences Γ to be inconsistent just in case we can derive a contradiction from it – i.e. $\Gamma \vdash \alpha \wedge \neg\alpha$ for some well-formed formula α. A set of sentences is consistent just in case it is not inconsistent. Similarly, we adopt the standard notion of soundness and say that a rule of inference is sound just in case, if $\Gamma \vdash \alpha$ then $\Gamma \vDash \alpha$. That is, if α is a deductive consequence of Γ, it is also a semantic consequence of it.

(CONSISTENCY) and (SOUNDNESS) are good candidates for cornerstones of ZFC. The reason is that both of them concern something integral to ZFC-theorizing:

Consistency is a necessary condition for a theory to be in good standing. Suppose that ZFC is inconsistent. Then there would be some α such that ZFC $\vdash \alpha \wedge \neg\alpha$. Now, it is a (derived) rule that anything follows from a contradiction, i.e. $\alpha \wedge \neg\alpha \vdash \beta$. So, if ZFC $\vdash \alpha \wedge \neg\alpha$ then ZFC $\vdash \beta$, for any statement β that can be formulated in the language of ZFC. In that case ZFC would prove the statement that every von Neumann ordinal is a transitive set well-ordered by $\in$, as it should. However, unfortunately, it would also prove the negation of this statement. Indeed, for *any* statement of the language of ZFC, ZFC would prove it and its negation. In other words, the inconsistency of ZFC would have a rather trivializing effect. Thus, in the absence of a warrant for the consistency of ZFC, we could not rationally claim warrant for belief in any statement proved from its axioms (and logic). The absence of such a warrant recommends doubt or at least open-mindedness about whether the theory is consistent, and, as a consequence, it would seem that belief in results established in the theory cannot rationally be claimed to be warranted. How could one rationally claim to acquire a warrant for a belief in results proved in ZFC, if, at the same time, one was open-minded as to whether the theory was trivial?

The soundness of the rules of inference of the background logic is also a good candidate for being a cornerstone of ZFC-theorizing. Because suppose that the rules of inference were not sound. Since soundness is what provides a guarantee that we are not led from truth to falsity when we reach the last line in a deduction, we could not in general rest assured to have established a proposition as true once it had been deduced from the axioms. In the absence of a warrant for the soundness of the logical apparatus behind ZFC, it would seem that we cannot in general rationally claim to acquire a warrant for a belief in the statement at the last step of a deduction in ZFC. As before, the absence of such a warrant suggests doubt or open-mindedness about the soundness of the logical machinery. However, how can one rationally claim a warrant to believe the statement at the last step of a proof, while being open-minded about the soundness of the logic relied on? Because belief is an attitude to the *truth* of a proposition, yet what one is open-minded about is exactly whether ZFC-deductions generally lead to truths.

So, it seems plausible to take the consistency of ZFC and the soundness of the rules of inference of the background logic as examples of cornerstones of ZFC.

3 Regress scepticism

Supposing that (CONSISTENCY) and (SOUNDNESS) are cornerstones of ZFC, it is important that they are warranted. If not, then, by the characterization of cornerstones, we cannot rationally claim a warrant for a belief in any ZFC-proposition! But are (CONSISTENCY) and (SOUNDNESS) warranted? And if so, how?

It might seem natural to suppose that all warrant is evidential. That is to say, whenever someone is warranted in believing some proposition, it is because the person has evidence which constitutes – or is intimately related to – the warrant. In mathematics proof is the paradigmatic example of evidence. It is often noted that proof is not just any kind of evidence, but *conclusive* evidence. Once something is proved, it stays that way. (This, of course, is compatible with someone mistakenly thinking that she has a proof of some theorem or proposition.) Though the centrality of proof in mathematics and logic cannot be denied, it better be the case that there are warrants that are not supplied by proof, provided that it is agreed that (CONSISTENCY) and (SOUNDNESS) are integral to ZFC in the sense of being cornerstones. At least there is a simple sceptical line of reasoning which purports to show that there is no way to come to acquire a warrant for these propositions through proof. The sceptical argument is a regress argument and runs as follows:

[SOUNDNESS]:
Suppose that we accept that there is an onus to justify the soundness of the logical rules of inference. How might one go about providing such a justification? Well, by proving soundness, of course! However, in proving the soundness of an inference rule in a given object language, we rely on a meta-theoretic version of that very rule. So, the proof raises another presupposition – namely the soundness of the meta-theoretic rule. Of course, we can continue to prove the soundness of the meta-theoretic rule, but this, in turn, will involve a meta-meta-theoretic version of the same rule ... and so forth.

[CONSISTENCY]:
Gödel's second incompleteness theorem tells us that, if T is (i) consistent, (ii) recursively axiomatizable, and (iii) powerful enough to express elementary arithmetic, then there is a statement $\mathrm{Con}(T)$ in the language of T stating the consistency of T which is not provable in T. By this theorem, no system strong enough to be mathematically interesting (i.e. elementary arithmetic) can prove its own consistency. So, e.g., the consistency of PA cannot be proved in PA itself, let alone any finite subsystem. In light of this, a common approach to the consistency question for a given mathematical theory T_1 is to try to prove T_1 consistent relative to another theory T_2, i.e. to show that

$$(*) \qquad \text{If } Con(T_2), \text{ then } Con(T_1).$$

This, however, invites a regress. Recall that what we are after is a warrant for a belief in $Con(T_1)$. One immediate observation is that, in case we appeal to $(*)$, the warrant for $Con(T_1)$ is held hostage to there being a warrant for $Con(T_2)$. However, a proof of $(*)$ does nothing to establish *that*. So we might bring yet another theory T_3 into play and show that

$$(**) \qquad \text{If } Con(T_3), \text{ then } Con(T_2).$$

It is not clear that this improves the situation, though. Things similar to those said about T_1 and T_2 can be said about T_2 and T_3, T_3 and T_4. And so forth.

On the basis of [SOUNDNESS] and [CONSISTENCY], the regress sceptic concludes that we cannot acquire a warrant for (SOUNDNESS) and (CONSISTENCY).

It is worth dwelling, if only briefly, on the regress reasoning. Part of the setup of [SOUNDNESS] and [CONSISTENCY] is that it is accepted that there is an onus to justify (SOUNDNESS) and (CONSISTENCY). Now, if the meta-theoretic rules of inference relied on in the proof of soundness were more secure than their object-theoretic analogues, the soundness proof would, arguably, have an epistemological pay off. Likewise, if T_2 was more secure than T_1, we would, arguably, have an epistemological pay off when proving T_1 consistent relative to T_2. Thus, if – at some point in the reasoning just rehearsed – one of the additional presuppositions brought into play was more secure than its 'predecessor', it would seem that the regress could be brought to a stop.

The regress sceptic, however, will respond that it is a feature of the regress reasoning that this cannot be the case. Consider [SOUNDNESS] first. The presuppositions brought into play – i.e. the soundness of the meta-theoretic inference rules – are of the exact same kind as the ones we set out to justify, and so, will be as epistemologically secure (or insecure, as the case may be) as the initial rules. As for [CONSISTENCY], the sceptic will point out that T_2 will be at least as epistemologically problematic as T_1, if not more. T_2 is more problematic than T_1 when it is strictly stronger than T_1. In that case the system brings into play not only presuppositions of the same kind as those of T_1, but also ones that push beyond them.

To illustrate, let *Inacc* be an abbreviation of the statement that there is a strongly inaccessible cardinal. Now, it is easy to show that ZFC is consistent if ZFC + *Inacc* is.[1] What can we legitimately take this result to show, by itself, about the epistemic status of the statement that ZFC is consistent? Nothing, really. In particular, the relative consistency result in question does not supply a warrant for a belief that ZFC is consistent. The consistency of ZFC is held hostage to that of ZFC + *Inacc*, which is stronger than ZFC itself.[2]

4 Lesson: some warrants do not require proof

I am not a regress sceptic. In fact, I doubt that anyone is. Yet it is worthwhile to reflect on regress scepticism. It can show us something important about mathematical warrant, *viz.* that there has to be mathematical warrants that do not require, or involve, proof. If we did indeed enforce such a requirement on warrant, we would be caught in the regress. However, it would be an epistemological catastrophe of sorts if we committed ourselves to a conception of mathematical warrant which implied that we could never acquire a warrant for (CONSISTENCY) and (SOUNDNESS): (CONSISTENCY) and (SOUNDNESS) are cornerstones of ZFC-theorizing, and, by the characterization of cornerstones, we could not rationally claim a warrant for *any* belief reached through ZFC-theorizing if we had no warrant for them. That would spell T-R-O-U-B-L-E – so we better not go there. The lesson learned: some warrants do not require proof.

5 Entitlement of cognitive project

As indicated above, the regress arguments suggest that proof can cater for a warrant for neither (CONSISTENCY) nor (SOUNDNESS). From this it follows that there can be no warrant for (CONSISTENCY) or (SOUNDNESS) only on the assumption that warrant based on, or constituted by, proof is the only kind of warrant there is. The previous section should make it clear that this assumption has been rejected. However, it is one thing to reject an assumption, and quite another to show how to meet the challenge, or challenges, raised by the rejection. Here the challenge is to characterize a notion of warrant which does not involve proof and which can be applied to (CONSISTENCY) and (SOUNDNESS). This is where Wright's notion of entitlement of cognitive project enters the scene.

The notion is characterized as folllows[3]:

> **Entitlement of cognitive project:** A proposition P is an entitlement of a cognitive project if
>
> (i) P is a *presupposition* of the project, i.e. if to doubt P (in advance) would rationally commit one to doubting the significance or competence of the project;
>
> (ii) we have no sufficient reason to believe that P is untrue; and
>
> (iii) the attempt to justify P would involve further presuppositions in turn of no more secure a prior standing ... and so on without limit; so that someone pursuing the relevant enquiry who accepted that there is nevertheless an onus to justify P would implicitly undertake a commitment to

> an infinite regress of justificatory projects, each concerned to vindicate the presuppositions of its predecessors.

As announced at the beginning of the paper, entitlement of cognitive project is a non-evidential kind of warrant. Clause (ii) is what makes it so. What is required for entitlement is not the presence of positive evidence, but rather the absence of sufficient countervailing evidence. Proof is one kind of evidence. Put specifically in terms of proof, what entitlement requires is not proof, but rather the absence of disproof. (We might choose to refer to clause (ii) as a 'default condition', because, given that conditions (i) and (iii) are satisfied, then the default position is that we have an entitlement to P. That is, P is entitled *unless* there is sufficient reason to believe that P is untrue.)

Clause (i) tells us that P is an unavoidable commitment or presupposition for engagement in the cognitive project C. Doubt about P brings with it a rational commitment to doubt about the competence of the project. Clause (iii) ensures that entitlements are among the most basic and secure propositions of the relevant region of thought. Attempts to justify an entitlement will never lead one to a proposition which is more secure.

The proposal that we want to present as a response to the regress sceptic and the challenge raised by our rejection of proof being requisite for mathematical warrant is, of course, the following:

> [THE ENTITLEMENT PROPOSAL]:
> (CONSISTENCY) and (SOUNDNESS) are warranted as a matter of entitlement.

For this to be a promising route to take it will obviously have to be the case that (CONSISTENCY) and (SOUNDNESS) satisfy clauses (i)-(iii) in the characterization of entitlement. Let us investigate the matter, bearing in mind that the relevant region of thought is ZFC-theorizing.

Are (CONSISTENCY) and (SOUNDNESS) presuppositions of ZFC-projects? They are. As argued above, both are cornerstones of ZFC, because they concern something integral to ZFC-theorizing. The question whether ZFC is consistent or not ties in with the question whether or not it is a worthwhile theory to work in. As seen above, inconsistency has a trivializing effect: if a theory T is inconsistent, it proves any statement α that can be formulated in the language of T *and* its negation. Speak of proving too much. For a given ZFC-project, doubt about (CONSISTENCY) thus brings with it a rational commitment to doubting the significance or competence of the project. Similar things can be said about (SOUNDNESS). To state the obvious: proofs involve logic. In particular, proofs in ZFC involve logic. However, if we want to prove something in ZFC, doubting the soundness of the standard rules of inference will rationally commit us to doubting the very project we want to undertake. Doubt about soundness amounts to doubt about whether applications of the rules really take us from truth to truth. However, harbouring such doubt we would be rationally committed to doubting the significance of the project (the proof might lead us to something false!). We conclude that both (CONSISTENCY) and (SOUNDNESS) are presuppositions of ZFC-projects.

Clause (ii) demands an absence of sufficient countervailing evidence. Let us first think in terms of proof. Is there an absence of a disproofs of respectively (CONSISTENCY) and (SOUNDNESS)? There is no proof that ZFC is inconsistent, at least none that we know of.[4] Likewise there is no proof that the rules of inference of classical logic – the background logic of ZFC – are not sound. Indeed, we have a proof that they are. However, as we have seen, the proof relies on the soundness of corresponding meta-theoretic versions of the rules.

Does it make a difference if we switch away from talk of proof and speak more generally about evidence? Clearly, it does so with respect to what we consider relevant to the task of assessing whether clause (ii) is satisfied or not, but not, I submit, with respect to the verdict in relation to (CONSISTENCY) and (SOUNDNESS). Just like I know of no proof of the inconsistency of ZFC, I know of no sufficient reason for thinking the theory false.

Some people might cite, say, the Banach-Tarski theorem as reason for thinking the axiom of choice false.[5] This is suggested by the fact that some refer to this theorem as the 'Banach-Tarski *paradox*'. One thing it is crucial to bear in mind in our present context is that a violation of clause (ii) requires not just any countervailing reason, but a *sufficient* one. So, although it might be granted that the Banach-Tarski theorem is a countervailing reason of some sort, it can still be denied that it is sufficient for thinking the axiom of choice – and so, ZFC – false. It might even be contested that it gives us a reason for thinking choice false at all. While the theorem might seem odd, or counterintuitive, given intuitions many people have about what can be done to solid bodies, it is not clear that this should count. Intuitions are not always reliable guides to what is true and what is false. (Given the intuitions many people have about space, it would seem that two parallel lines can never meet. However, do we take that to count as a reason for thinking non-Euclidean geometries false?)

What about clause (iii) – do attempts to justify (CONSISTENCY) and (SOUNDNESS) give rise to an infinite regress? It would seem so. At least this is what our considerations on regress scepticism suggest. Attempts to justify each of them brings up further presuppositions of a no more secure standing than the presuppositions already in play. In the case of soundness the further presuppositions were the soundness of rules of inference of the very same kind as those whose soundness we set out to justify, and in the case of consistency, the further – possibly *less* secure – presuppositions were embedded in the theory to which the initial theory was proved relatively consistent.

So, it seems like (CONSISTENCY) and (SOUNDNESS) satisfy clauses (i)-(iii) of the characterization of entitlement of cognitive project. This is good news, I think – being, as I am, a friend of the entitlement proposal.

6 Conclusion

The aim of this paper was to discuss the question whether the notion of entitlement of cognitive project can do any work in the philosophy of mathematics. Having offered some considerations on regress scepticism, it was suggested that the notion offers a way out of the regress pressed by the sceptic. We blocked the regress by taking issue with – or stronger: rejecting – the implicit assumption that every warrant involves proof. However, since such a rejection is entirely negative in character, we still had to say something positive about what kind of warrant might be appropriate to account for cornerstone warrant. Entitlement of cognitive project was introduced for this purpose.

Notes

[1] A cardinal κ is strongly inaccessible just in case: (i) $\kappa > \aleph_0$, (ii) $2^\lambda < \kappa$ for any $\lambda < \kappa$, and (iii) κ cannot be represented as the supremum of fewer than κ smaller ordinals. (i) ensures that κ is uncountable. (ii) and (iii) ensure closure under the operations of power set and union. The idea behind the relative consistency proof is roughly this: consider any model V_λ of ZFC + *Inacc*. By *Inacc*, there is a strongly inaccessible cardinal in V_λ – let it be κ. Consider V_κ, an initial segment of V_λ. It is straightforward to verify that the axioms of ZFC are satisfied when their quantifiers are restricted to V_κ.

[2] Klaus Frovin Jørgensen and others have reminded me of Gentzen's consistency proof for arithmetic. The proof relies crucially on a restricted form of transfinite induction (up to ϵ_0, to be specific). The regress sceptic might insist that the use of this kind of induction brings into play presuppositions which are more problematic from an epistemological perspective than those of arithmetic itself. Here it is worth noting that Gentzen himself says, 'A consistency proof can merely *reduce* the correctness of certain forms of inference to the correctness of other forms of inference' – and continues, 'It is therefore clear that in a consistency proof we can use only forms of inference that count as considerably *more secure* than the forms of inference of the theory whose consistency is to be proved.' (Gentzen [1], p. 138.) What is interesting in our present context is Gentzen's remark that the rules of inference relied on in a consistency proof have to be '*more secure*' than those of the target theory. It is beyond the scope of this short paper to discuss Gentzen's proof any further, but – for what it is worth – let me suggest that the proof *is* interesting from a philosophical perspective and apologize to the reader for not being able to discuss it extensively here.

[3] Wright [2], pp. 191–192.

[4] Though, occasionally, people have claimed to have such a proof. One example is Bryan Ford's alleged proof of a result that would imply the inconsistency of ZFC. See the FOM mailing list for details.

[5] The theorem states that it is possible to divide a solid ball into six parts which, by rigid movement, can be combined into two balls each of which is the size of the original.

References

[1] G. Gentzen: 'The Consistency of Elementary Number Theory', pp. 132–213 in M. Szabo (ed.): *The Collected Papers of Gerhard Gentzen* (Amsterdam: North-Holland), 1969.

[2] C. Wright: 'Warrant for Nothing (and Foundatinos for Free)?', pp. 167–212 in *Proceedings of the Aristotelian Society Supplementary Vol. LXXVIII*, 2004.